An ϵ-expansion for Small-World Networks

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Abstract. – I construct a well-defined expansion in $\epsilon = 2 - d$ for diffusion processes on small-world networks. The technique permits one to calculate the average over disorder of moments of the Green's function, and is used to calculate the average Green's function and fluctuations to first non-leading order in ϵ, giving results which agree with numerics. This technique is also applicable to other problems of diffusion in random media.

The small-world network [1] has served as a fundamental model in the field of networks [2]. However, the problem of averaging over the possible different random connections in the small-world network is severe in low dimensions: a study [3] of the properties of even the simple problem of diffusion on the one-dimensional network leads to a difficult problem that, thus far, has only been tackled approximately.

The physical reason for this problem is a breakdown of mean-field theory [4] in dimensions d less than two, and the emergence of strong site-to-site fluctuations of the Green's function, so that the properties of the system cannot be represented by simply studying the average. However, this opens the possibility of perturbing in $\epsilon = 2 - d$, as will be shown in this paper.

The small-world network is constructed by starting with a regular lattice in d-dimensions. Then, some set of long-range links are added: a given pair of sites i, j is connected with probability pa^d/V, where V is the total number of sites in the system. Here, we define a length a as the lattice scale, and p as the *density* of links. Then, as $V \to \infty$, each site has a Poisson distribution of links emanating from it, with on average pa^d links. Typically, the links, if any, leaving a given site will connect that site to other sites far away in the system.

Looking for universal results, we consider the case of a low density of links, $pa^d << 1$. Ignoring sample-to-sample fluctuations, the natural mean-field system to consider is one in which each site is coupled to all others with a strength $\sim p/V$. This leads to a solvable problem with a correlation length $\xi \propto p^{-1/2}$, or a correlation volume $\xi^d \propto p^{-d/2}$. Returning to the original problem with fixed links, we see that such a volume has $p^{1-d/2}$ links in it, and as $p \to 0$, this number of links tends to zero for $p \leq 2$. This is a major problem. Mean-field theory ignores fluctuations in the number of links, which is only justified if the number of links is large, a condition which is not satisfied in this case for $d < 2$. This problem is a result of a violation of the modified Harris criterion introduced in [5].

Instead, we expect that the correlation length for the average Green's function must be at least $p^{-1/d}$ for small p, so that there is on average at least one link in a correlation volume. We

will see below that the correlation length is in fact proportional to $p^{-1/d}$ in this limit, and the ϵ-expansion will enable us to calculate the prefactor, as well as to study fluctuations about the average. Given a $p^{-1/d}$ scaling of the correlation length, the number of links in a correlation volume is some p-independent number. The basis for the ϵ-expansion is the observation that, within a self-consistent approximation, this number diverges as $1/\epsilon$, so that fluctuations about a self-consistent mean-field can be calculated diagrammatically as done below.

Green's Function— Our general problem is to study diffusion on the small-world network. Thus, we must calculate Green's functions of the following problem: $\partial_t \rho_i(t) = -\sum_j \Gamma_{ij}\rho_j$, where $\rho_i(t)$ is the probability of finding some randomly walking particle at site i and Γ is the Laplacian on the small-world network. We take

$$\Gamma = \Gamma^0 + qU. \tag{1}$$

Here, Γ^0_{ij} is the Laplacian on the regular part of the network. $\Gamma^0_{ij} = -a^{-2}$ if i and j are neighboring sites on the regular network, while Γ^0_{ii} is equal to a^{-2} times the coordination number of site i. U_{ij} is the Laplacian on the long-range links. $U_{ij} = -1$ if i, j are connected by a long-range link, where U_{ii} is equal to the number of long-range links leaving site i.

We have inserted an extra factor of q multiplying the matrix U in Eq. (1). If q is small and p is large, this implies that we have a high density of weak links and the problem can be solved via a mean-field theory in which we ignore the fluctuations in the local density of links. However, we will be interested in the opposite case, where q is of order 1, while p is small.

Fourier transforming, we are interested in the Green's function $(i\omega + \Gamma)^{-1}$. However, we will focus on the case $\omega \to 0$. The matrix Γ has a zero single mode, due to the conservation of $\sum_i \rho_i$ by the diffusion process. Throughout, we will work in the subspace orthogonal to this zero mode, defining $G = \lim_{\omega\to 0}(i\omega + \Gamma)^{-1}$ and $G^0 = \lim_{\omega\to 0}(i\omega + \Gamma^0)^{-1}$ in this subspace. This Green's function is related to the return probability of a random walker, and also to the roughness of a surface defined by Edwards-Wilkinson dynamics on the network [4]. We will compute $\overline{G_{ij}}$, where the line denotes averaging over the ensemble of different random networks, as well as computing higher moments such as $\overline{G_{ij}G_{kl}}$.

For p small (compared to a^{-d}), the length $p^{-1/d}$ is much larger than the lattice scale, so that we can take a continuum limit, setting Γ^0 equal to the continuum Laplacian ∂^2. In the limit of small p, the probability of a single site having more than one link becomes vanishingly small. This continuum limit leads to ultraviolet divergences for $d \geq 2$, which are cutoff at the lattice scale a. However, for $d < 2$, this limit is completely convergent, and thus for $d < 2$ we find universal results, independent of the lattice details. This use of a continuum limit is essential to continue the results to arbitrary real dimension $d < 2$.

For clearer notation, in the continuum limit we will often use d-dimensional vectors $x, y, ...$ to label lattice sites, rather than indices $i, j, ...$ We write $G(x, y)$ to denote a Green's function between points x, y and $\overline{G(x)}$ to denote a Green's function $\overline{G(x, 0)}$ (the averaging restores translational symmetry). We also use a vector k to label momenta, defining the averaged Green's function at Fourier mode k to be $\overline{G(k)} = \int \mathrm{d}^d x \overline{G(x)} \exp(ik \cdot x)$. We use a similar notation for other matrices in the continuum limit. Discrete δ-functions get replaced by Dirac δ-functions, while sums get replaced by integrals.

Ignoring fluctuations in the local density of links, the mean-field approximation consists of replacing U by $\overline{U}$, so in the continuum limit $\overline{G} \approx (\partial^2 + \overline{U})^{-1}$. Then, we find that $\overline{U(x, y)} = pq\delta(x - y) - pq/V$. This gives a Green's function $\overline{G(x)} = \int \mathrm{d}^d k (2\pi)^{-d} \exp(ik \cdot x)(k^2 + pq)^{-1}$. The correlation length is $(pq)^{-1/2}$ as discussed above, and thus this expansion must breakdown for $d < 2$ in the limit of small p at fixed q.

Self-consistent Calculation— To go beyond the mean-field calculation, we use impurity-averaged perturbation theory [6], following [4]. We continue with lattice notation here, since

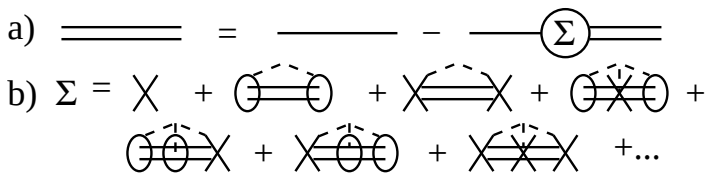

Fig. 1 – a) Diagrammatics for Σ and b) Self-consistent sum of interactions with a single link.

this development is not specific to the continuum limit. The perturbative expansion of $\overline{G}$ in powers of U is $\overline{G} = G^0 - \overline{G^0 U G^0} + \overline{G^0 U G^0 U G^0} - ...$, where a product of matrices is implied. Each of the terms in this expansion can be computed using the given distribution of disorder. The mean-field calculation is based on the following approximation: $G \approx (\Gamma^0 + \overline{U})^{-1} = G^0 - G^0 \overline{U} G^0 + G^0 \overline{U} G^0 \overline{U} G^0 - ...$ These two expansions are equal at zeroth and first order in U, but differ at second order in U, by an amount depending on the second cumulant of U.

It is useful to introduce a diagrammatic notation for the perturbative calculations, as shown in Fig. 1. We use a single solid line to denote G^0, and a double solid line to denote $\overline{G}$. We use a cross with no dashed lines attached to denote the average $\overline{U_{ii}}$. A pair of crosses connected by a dashed line is used to denote an average $\overline{U_{ii} U_{jj}} - \overline{U_{ii}}\ \overline{U_{jj}}$. Three or more crosses connected by dashed lines are used to denote the third and higher cumulants. These diagrams denote two, three, or more scatterings of a single link. Circles are used to denote averages of off-diagonal terms, U_{ij} for $i \neq j$. Dashed lines can also connected both circles and crosses, again denoting higher cumulants. Within this diagrammatic approximation, the weight of a given diagram is equal to $p^{n_l}(-q)^{n_s}$. Here n_l is equal to the number of sets of circles or crosses connected by dashed lines, while n_s is equal to the total number of circles plus crosses. Thus, each link gives one factor of p (the probability of finding a link) multiplied by $-q$ to the number of times that link appears in U.

We note that a single circle, not connected by a dashed line, denotes $\overline{U_{ij}}$, which vanishes as $V \to \infty$, and thus may be ignored in this perturbation expansion. In general, any diagram involving an odd number of off-diagonal terms for a given link vanishes as $V \to \infty$.

Let us return to the continuum limit for specific calculations. We introduce the self-energy $\Sigma(k)$ by setting $\overline{G(k)} = [\Gamma^0 + \Sigma(k)]^{-1}$, as shown diagrammatically in Fig. 1(a). Approximating Σ to leading order in U we get $\Sigma(k) = \overline{U(k)}$. We have $\Sigma(0) = 0$, However, for all $k \neq 0$, $\Sigma(k)$ is k-independent, so that, outside of the subspace of the zero mode, we may write $\Sigma(k) = \Sigma = pq$. We now compute corrections to this result, to second order in U. To this order, $\Sigma(k)$ is still k-independent for $k \neq 0$. This self-consistent Born approximation is defined by the first three diagrams on the right-hand side of Fig. 1(b). Higher diagrams of Fig. 1(b) include all diagrams with arbitrary numbers of scatterings off a *single* link. We self-consistently use double lines for the Green's function in Fig. 1(b), giving $\Sigma = pq - 2pq^2 \overline{G(0)}$, where

$$\overline{G(0)} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 + \Sigma}. \tag{2}$$

For $d < 2$, the integral of Eq. (2) is convergent, and equal to

$$\frac{\pi^{d/2}}{(2\pi)^d} \Sigma^{d/2-1} \Gamma(1 - d/2). \tag{3}$$

The self-consistent Born approximation at second order in U then becomes $\Sigma = pq - 2pq^2 \overline{G(0)}$. As a first guess at solving this equation, we substitute the first order result for Σ in the equation

for $\overline{G(0)}$, getting $\Sigma = pq - 2p^{d/2}q^{d/2+1}\pi^{d/2}\Gamma(1-d/2)/(2\pi)^d$ for $d < 2$. The second order correction is comparable to the first when $q \sim p^{2/d-1}$. Thus, for $q << p^{2/d-1}$, perturbation theory may be applied. We will be interested in the opposite limit of p small at fixed q so that $q >> p^{2/d-1}$ and perturbation theory breaks down for $d < 2$.

A better approximation for Σ is to include all diagrams in Fig. 1(b), summing all diagrams involving interactions with a single link. Since we are interested in the case in which the density of links is small, but the scattering q off a single link is of order unity, we sum all scatterings off a single link. This gives $\Sigma = pq/[1+2q\overline{G(0)}]$. For p small, we will find $\overline{G(0)} >> 1$, so that we can take $\Sigma = pq/[2q\overline{G(0)}] = p/[2\overline{G(0)}]$. Here, the factor of two arises from the different possible diagonal and off-diagonal scatterings off a single link: if a particle interacts with a given link n times, then there are 2^{n-1} total diagrams contributing to Σ. Using Eq. (3) for $\overline{G(0)}$, we find for $d < 2$ that the solution of $\Sigma = p/[2\overline{G(0)}]$ is $\Sigma =$

$$\Sigma_0 = [(2\pi)^d p/(2\pi^{d/2}\Gamma(1-d/2)]^{2/d}. \tag{4}$$

We have placed the subscript 0 on Σ, since later this result will be used as a zero order approximation, with corrections to it in powers of ϵ. This gives $\Sigma \propto p^{2/d}$, so that the correlation length varies as $p^{1/d}$. We will find below that a more careful treatment leads to universal corrections in orders of ϵ to the prefactor of Eq. (4), but does not change the scaling with p.

In contrast, for $d > 2$, the integral of Eq. (2) is divergent at large k, but it converges at small k even for $\Sigma = 0$. The divergence at large k is cut off by the lattice scale, a. Thus, $\overline{G(0)}$ has a well-defined limit as $\Sigma \to 0$; this limit has a non-universal dependence on the lattice details and is equal to some number, g. The self-consistent equation becomes $\Sigma = pq/[1+2q\overline{G(0)}]$. To leading order in p, the solution of this equation gives $\Sigma = pq/(1+2qg)$. The magnitude of the corrections to the mean-field result, $\Sigma = pq$, depends on the product qg and does not have any universal behavior. There are also further corrections to the mean-field result which involve scattering off multiple links. These corrections will not be considered here.

Finally, consider $d = 2$. In this case, we will show below that the approximation above leads to *exact* results for the small p behavior of Σ. The integral of Eq. (2) is logarithmically divergent. The divergence is cut off at a k of order a^{-1}, giving $\overline{G(0)} = -(2\pi)^{-1}\log(a\Sigma^{1/2})$. The self-consistent equation then gives $\Sigma = pq/[1+2q\overline{G(0)}]$. As for $d < 2$, $G(0) >> 1$ for small p so that this reduces to $\Sigma = p/[2\overline{G(0)}] = -2\pi p/\log(a^2\Sigma)$. Solving this self-consistent equation for $a^2 p << 1$ gives $\Sigma = -2\pi p/\log(pa^2)$, plus subleading terms.

Corrections in Two Dimensions and Renormalization Group— Thus far, the calculation has followed [4]. Now, however, we go beyond the self-consistent calculation to obtain an expansion in ϵ. The first step is to analyze the behavior for $d = 2$ in more detail, and show that the results above are exact for the leading scaling of Σ with p. In the next section, a diagrammatic perturbation expansion will be presented to compute results in powers of ϵ.

In $d = 2$, the correlation volume is of order $1/\Sigma$. The physical motivation for the results below is that for $\Sigma \sim -p/\log(pa^2)$, the average number of links in a correlation volume, p/Σ, is of order $-\log(pa^2)$. Thus, the average number of links in a correlation volume diverges as $p \to 0$. This, however, is exactly the condition we need to make mean-field theory work, as it enables us to ignore fluctuations in the number of links.

Turning away from perturbation theory, we now consider the problem scale-by-scale, starting with the shortest distances. We begin with a physical description of the problem, followed by a more careful calculation below. On length scales $l << p^{-1/d}$, each link can be considered independently, since the density of links at such scales is very small: the probability of finding a link near any other link is negligible. Solving the problem of a single link at a scale l is very similar to the calculation done above. We must sum multiple scatterings off the single

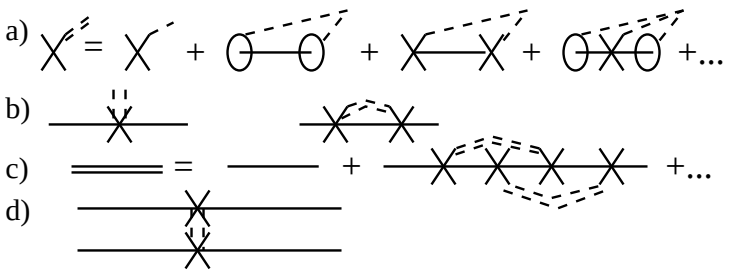

Fig. 2 – a) Definition of resummed scattering. b) Disallowed diagrams. c) Leading corrections to $\overline{G}$. d) Leading contribution to $\overline{GG} - \overline{G}\,\overline{G}$. For all diagrams, single line denotes G^l in this figure.

link. This will renormalize the interaction strength of the link from q to $q/[1 + 2qG_l(0)]$, where $G_l(0) \sim (2\pi)^{-1} \log(l/a)$ is the Green's function of matrix Γ^0 at scale l. Any function which cuts off momenta less than l^{-1} will suffice for G_l. We choose $G_l(k) = (k^2 + l^{-2})^{-1}$. Then, once we reach scale $l = p^{-1/2}$, the link interaction strength has been renormalized to $q_{ren} = q/[1 + 2qG_l(0)] \sim -2\pi/\log(pa^2)$.

The number of links in an area of size a^2 was pa^2. In contrast, on the new scale $l = p^{-1/2}$, the number of links in area l^2 is of order unity, while the scattering strength q_{ren} is small, and thus corrections to mean-field theory become negligible for small p. Then, we find $\Sigma = pq_{ren} = -2\pi p/\log(pa^2)$.

Expansion in ϵ— Now, we wish to compute quantities for $d < 2$. In this case, we find that once we reach a scale $l \sim p^{-1/d}$, the link interaction strength q_{ren} has been reduced to an amount of order $q/[1 + 2qG_l(0)] \sim \epsilon l^{-\epsilon}$. Unlike the case in two dimensions, q_{ren} does not vanish as we take the limit $pa^d \to 0$. However, we will find that the interaction strength of the links is of order ϵ, so that corrections to mean-field theory can be computed in powers of ϵ. The ϵ-expansion is defined by a simple re-ordering of diagrams. First, note that Eq. (4) defines a length scale l by

$$l = \Sigma_0^{-1/2} = [(2\pi)^d p/(2\pi^{d/2}\Gamma(1 - d/2)]^{-1/d}. \tag{5}$$

so $G_l(k) = (k^2 + \Sigma)^{-1}$. Then, we define the resummed diagonal scattering off a single link as in Fig. 2(a), denoted by a cross with double dashed lines. Here, the single line denotes the Green's function G_l, rather than G^0 as before (this resummation is closely related to the T-matrix in scattering theory). The resummed off-diagonal scattering is defined similarly by a circle with double dashed lines.

The starting point for the ϵ-expansion is the self-consistent calculation of Eq. (4). That expansion resums a large set of diagrams known as the "rainbow" diagrams; this is the set of all diagrams except those in which a particle first scatters off one link, then off a second, then returns to the first link, and then returns to the second link (possibly with additional scatterings off other links included). We now present an expansion for $\overline{G}$ which exactly includes all of the missing diagrams, such as in Fig. 2(c), with the correct coefficent, using resummed scatterings with each link. We then show that this leads to an expansion in powers of ϵ.

First, write down all possible diagrams, using G_l for the Green's functions and double dashed lines for interactions with links, subject to the following two constraints: (1) double dashed lines must always connect at least two crosses or circles. That is, single crosses or circles never appear individually, so the first diagram of Fig. 2(b) is not allowed. This constraint on diagrams is included because such diagrams are already taken into account in

the self-consistent expansion, by the replacement of G^0 by G_l. (2) two successive interactions with links must always involve different links (links not connected by dashed lines), so that the second diagram of Fig. 2(b) is also not allowed. The second constraint is presented to avoid overcounting multiple scatterings. Given these constraints, the leading correction to the averaged Green's function is given by Fig. 2(c). Each diagram is assigned a factor of $p^{n_l}(-q_{ren})^{n_s}$, where n_l is now the number of sets of circles and crosses connected by double dashed lines and n_s is again the total number of circles and crosses. Here, $q_{ren} = q/[1 + 2qG_l(0)] = \Sigma_0/p$.

This expansion, though it simply reorders the diagrams, perfectly realizes the desired treatment of the problem scale-by-scale. We must start with the shortest scales. However, at the shortest scales, the only processes involve multiple scattering off single links; only when the scale becomes of order $p^{-1/d}$ do processes with multiple links become important. Thus, we resum scattering off the single link, up to the scale l. Unlike the case for $d = 2$, processes involving multiple links, such as that shown in Fig. 2(c), are no longer vanishing, but we now show that they are higher order in ϵ. First, it is convenient to rescale all distances by l. Then, we set $\tilde{x} = x/l$, $\tilde{k} = kl$. We use the Green's function $G_1(k) = (k^2 + 1)^{-1}$, and define $\tilde{p} = pl^d$, while $\tilde{q}_{ren} = q_{ren}l^{2-d}$. Then, we define $\overline{\tilde{G}(\tilde{x})}$ by $\overline{G(x)} = \overline{\tilde{G}(\tilde{x})}l^{2-d}$. Then, the perturbation expansion for $\overline{\tilde{G}}$ is obtained by using the same set of diagrams as above, but replacing p by $\tilde{p}$, q_{ren} by $\tilde{q}_{ren}$, and G_l by G_1. Now, we have

$$\tilde{p} = 2\pi^{d/2}\Gamma(1 - d/2)/(2\pi)^d = \epsilon^{-1}4\pi^{d/2}/(2\pi)^d + ..., \qquad (6)$$
$$\tilde{q}_{ren} = \tilde{p}^{-1} = \epsilon(2\pi)^d/(4\pi^{d/2}) + ...,$$

where here we have given a series expansion of the results for $\tilde{p}, \tilde{q}_{ren}$, and the ... denote higher order terms in ϵ. Thus, we have defined a new problem of scattering with links of strength ϵ and density ϵ^{-1}; this a problem of a high density of weak links so that perturbative techniques work well and lead to an expansion in ϵ. Physically, one can imagine that for d close to two, a randomly diffusing particle has only a weak interaction with a link: the dimension of the path of the particle is two, so that for d close to two the particle can easily "miss" a given link.

There are a finite number of diagrams at each order in ϵ as we now show. The number of scatterings of each links is at least two by the rules above, so $n_s \geq 2n_l$. Then, $\tilde{p}^{n_l}\tilde{q}_{ren}^{n_s} \sim \epsilon^{n_s - n_l}$ is at least order ϵ^{n_l}. Thus, to order ϵ^n, we need only consider diagrams $n_l \leq n$. For a given n_l, we need only consider diagrams with $n_s \leq n + n_l$, leaving us with only a finite number of diagrams at each order in ϵ. It is important to consider the possibility of ultraviolet divergences in two dimensions, since they may give extra factors of ϵ^{-1}. However, the only ultraviolet divergences in two dimensions in this expansion arise from Green's functions $G_1(0)$ and this expansion resums all such divergent diagrams which involve scattering off at least one link: the only divergent contribution to $\overline{G(0)}$ is from the very first diagram in Fig. 2(c). Finally, to all orders in ϵ, the scaling of $\overline{G(0)}$ with p is unchanged from the self-consistent calculation.

We now use this formalism to compute specific results. We first compute the average Green's function. From Fig. 2(c), we have $\overline{G(0)} = l^{2-d}[G_1(0) + \tilde{p}^2\tilde{q}_{ren}^4 \int\int \mathrm{d}^d x \mathrm{d}^d y G_1(x)G_1(y)G_1(x-y)^3 + ...]$. We have numerically evaluated this integral in $d = 2$ to get the leading corrections in ϵ (to simplify this integral, we used the trick that it equals $-(1/4)\partial_l \int \mathrm{d}^2 x G_l(x)^4$ at $l = 1$). The result is $\overline{G(0)} = l^{2-d}[G_1(0) + \epsilon^2\pi^2(.00106...) + ...]$, where we have also evaluated $\tilde{p}, \tilde{q}_{ren}$ for $d = 2$, and where l is given by Eq. (5). Here, $G_1(0) = \pi^{d/2}/(2\pi)^d\Gamma(1 - d/2)$ and hence diverges as ϵ^{-1}. As noted above, this is the only ultraviolet divergent diagram.

To go to higher order in ϵ, it would be necessary to keep additional diagrams, as well as to expand the integrals in powers of ϵ near $d = 2$. Since we are interested in $d = 1$, let us first

evaluate Fig. 2(c) in $d = 1$, where $\tilde{p} = \tilde{q}_{ren} = 1$, so

$$\overline{G(0)} = p^{-1}[G_1(0) + \int\int \mathrm{d}x\mathrm{d}y G_1(x)G_1(y)G_1(x-y)^3 + ...] = p^{-1}(1/2 + 5/256 + ...). \quad (7)$$

At higher order, if we evaluate all diagrams in $d = 1$, but use the ϵ expansion to define the ordering of diagrams for us, all the diagrammatic integrals can be performed exactly, since $G_1(x)$ then has a simple exponential decay. This is a task left for future work, as is a test of the convergence properties of this expansion. Eq. (7) compares well with numerics [4]. There, $\overline{G(0)}$ was found to scale p^{-1}, with a prefactor slightly larger than $1/2$, close to the result here; comparison of the exact difference between the prefactor and $1/2$ would require higher order calculations here and larger system sizes in the numerics.

Fluctuations in the Green's function can also be computed using these techniques. From Fig. 2(d), we have $\overline{G(0)^2} - (\overline{G(0)})^2 = p^{-2}(\int \mathrm{d}x G_1(x)^4 + ...) = p^{-2}(1/32 + ...)$

Discussion— We have presented an ϵ-expansion for the properties of the small-world network in $d = 2 - \epsilon$ dimensions, enabling us to compute averages of moments of the Green's function. This technique should, however, have much greater generality. In [5], general criteria were put forth for the breakdown of mean-field theory for any statistical mechanical system on a small-world network, in analogy to the usual Harris criterion for disordered systems [7]. Hopefully, for other systems in which these criteria are violated, it will be possible to provide an ϵ expansion near the critical dimensionality, analogous to what has been done for the violation of the ordinary Harris criterion [8].

These criteria can be extended to other problems than networks, such as a randomly diffusing particle in an array of traps [9], a model inspired by work in reaction-diffusion processes [10]. The criteria [5] correctly predict $d = 2$ as the region for the breakdown of mean-field theory, and the techniques here can be used to compute average properties of the Green's function in $d = 2$. For the random trap system in $d = 1$ with $p << q$, the particle has only a small probability to pass through one trap to reach the next trap which is typically far away. This makes the trap system very far from mean-field theory and suggests that the ϵ-expansion will not converge to $\epsilon = 1$; however, it also enables the solution of the $d = 1$ system by considering each interval between traps independently. In contrast, for the network, the particle can interact with several links, even for $p \to 0$, thus suggesting that the perturbative ϵ-expansion may work for the small-world case. A true test will require evaluation of additional higher order diagrams and is left for future work.

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